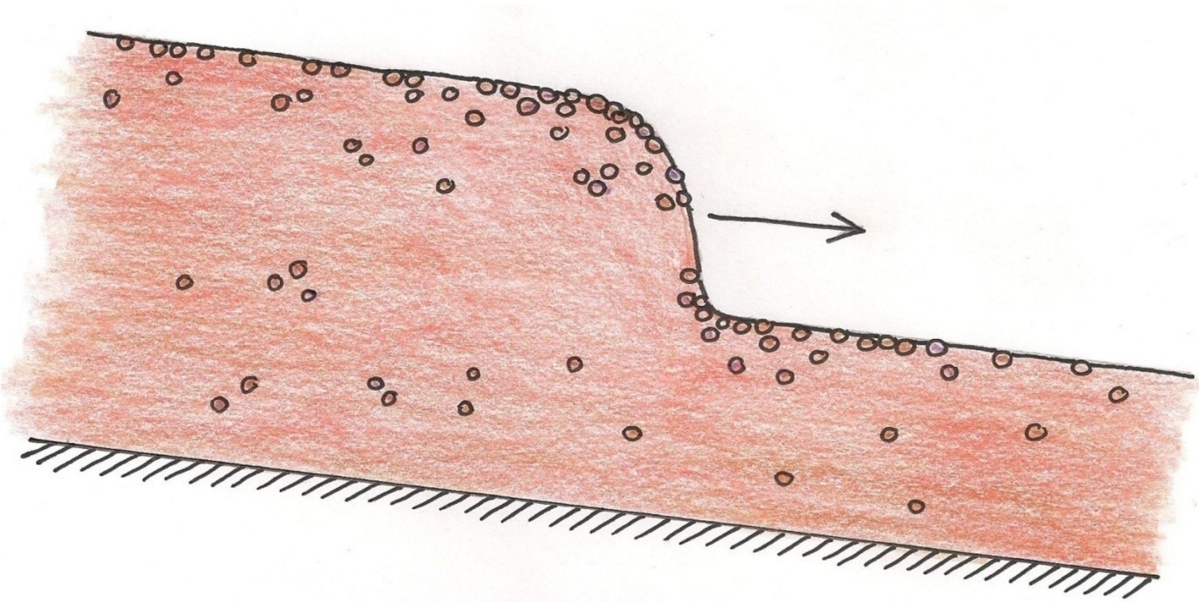
OK

The Granular Monoclinal Wave



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Monoclinal flood wave in water

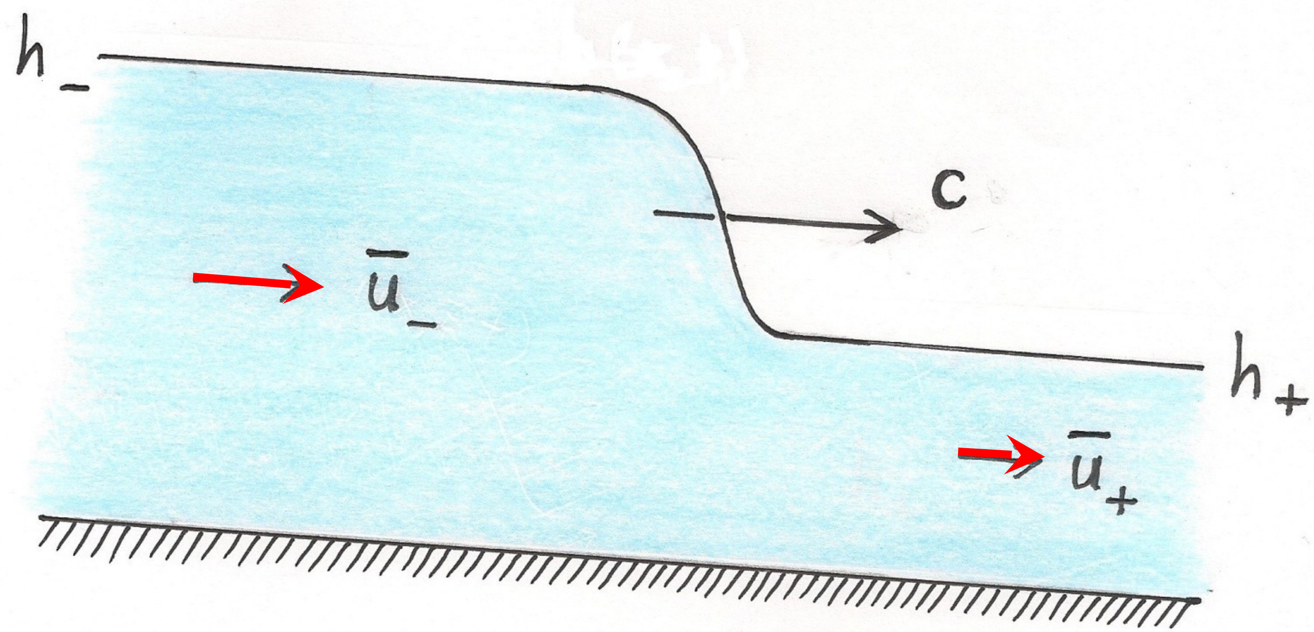
well described by the Saint-Venant equations, which are two coupled PDEs for

$h(x,t)$ and $\bar{u}(x,t)$

height of the water sheet

depth-averaged velocity

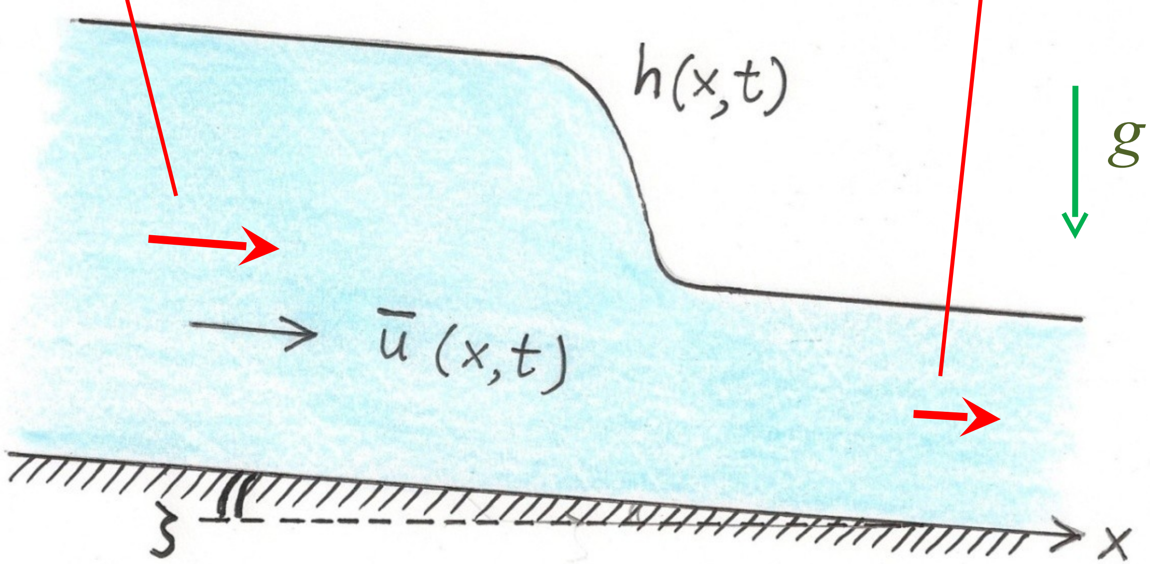
Velocity profile:



Depth-averaged velocity:

$$\boxed{\bar{u}(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} u(x,z,t)\, dz}$$

$$\boxed{\mathrm{Fr}_{in} = \frac{\bar{u}_-}{\sqrt{h_- \, g \cos\zeta}}} \quad > \quad \boxed{\mathrm{Fr}_{downstream} = \frac{\bar{u}_+}{\sqrt{h_+ \, g \cos\zeta}}}$$

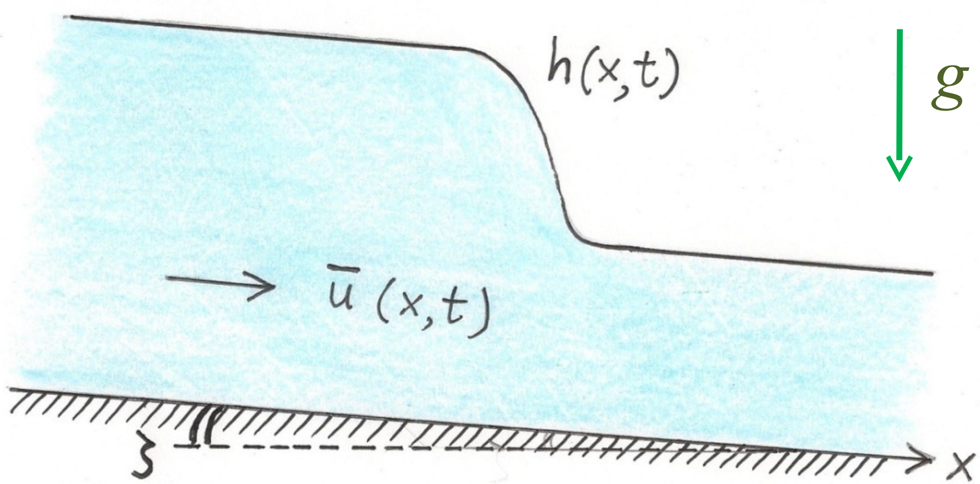


Another important concept in this context: the Froude number

inertial forces vs. gravity

$$\mathrm{Fr} = \mathrm{Fr}(x,t) = \frac{\bar{u}(x,t)}{\sqrt{h(x,t)\, g \cos\zeta}}$$

The Saint-Venant equations



1. Conservation of mass:
(continuity equation)

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0$$

2. Momentum balance:

$$\underbrace{\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(h\bar{u}^2)}_{\text{momentum change}} = hg\sin\zeta - \frac{\partial}{\partial x}\left(\tfrac{1}{2}gh^2\cos\zeta\right) + \text{terms involving friction and viscosity}$$

momentum change

gravity component in x-direction

gradient of depth-averaged pressure

In fact, the Saint-Venant equations admit various types of waves:

also known in granular flow

not yet observed in granular flow

also in granular flow

Gray & Edwards, JFM **755** (2014)

also in granular flow

Boudet *et al*., JFM **572** (2007)

Now we turn to granular matter

sand avalanches, landslides, etc.

Saint-Venant equations for granular flow

Mass conservation: $$\partial_t h + \partial_x \left(h\overline{u} \right) = 0$$

Momentum balance:

$$\partial_t \left(h\overline{u} \right) + \partial_x \left(h\overline{u}^2 \right) =$$

$$gh\sin\zeta - \tfrac{1}{2}\partial_x \left(gh^2 \cos\zeta \right) - \mu(h,\overline{u})\, gh\cos\zeta + \nu\, \partial_x \left(h^{3/2} \partial_x \overline{u} \right)$$

special granular features:

friction with the chute

viscous-like term

Special granular features 1: the friction law

$$\partial_t\left(h\bar{u}\right)+\partial_x\left(h\bar{u}^2\right)\;=$$

the friction coefficient $\mu(h,\bar{u})$ depends on h and $\bar{u}$

$$g h\sin\zeta-\tfrac{1}{2}\partial_x\left(g h^2\cos\zeta\right)-\boxed{\mu(h,\bar{u})\,g h\cos\zeta}+\nu\,\partial_x\left(h^{3/2}\partial_x\bar{u}\right)$$

And also on the inclination ζ of the chute:

- In the case of **water**, an arbitrarily small *shear stress* is sufficient to put the fluid in motion.

- **Sand**, however, remains motionless until the *shear stress* exceeds a certain threshold value => so the slope of the chute ($\boldsymbol{\zeta}$) has to exceed a critical value $\boldsymbol{\zeta_1}$ in order to make the sand flow. On the other hand, the slope must stay below a second critical value $\boldsymbol{\zeta_2}$ because otherwise the sand will rush down in an uncontrollable avalanche

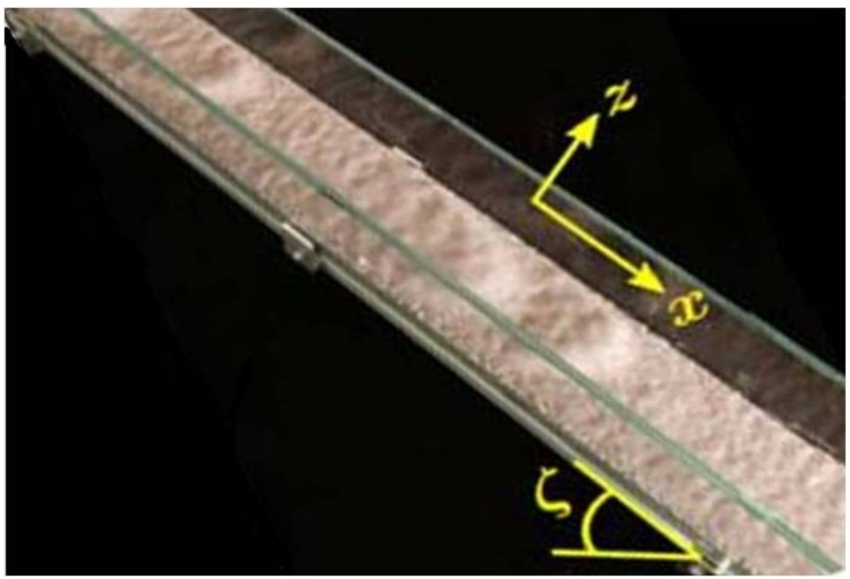
Only in the intermediate interval $\boldsymbol{\zeta_1 < \zeta < \zeta_2}$ we can witness the balance between forces necessary for a steady flow.

Friction coefficient

Pouliquen and Forterre, J. Fluid Mech. **453** (2002)

$$\mu\left(h,\bar{u}\right) = \tan\zeta_1 + \frac{\tan\zeta_2 - \tan\zeta_1}{1+(\beta/L)\,\dfrac{h^{3/2}}{\bar{u}}\sqrt{g\cos\zeta}}$$

(for Fr > β)



In our system:

$$\zeta_1 = 32.9^\circ$$

$$\zeta = 33.3^\circ$$

$$\zeta_2 = 42.0^\circ$$

$$\beta = 0.5$$

$$L = 1\ \text{mm}$$

Special granular features 2: the viscous term

$$\partial_t(h\bar{u}) + \partial_x(h\bar{u}^2) =$$

$$gh\sin\zeta - \tfrac{1}{2}\partial_x\left(gh^2\cos\zeta\right) - \mu(h,\bar{u})gh\cos\zeta + \boxed{\nu\,\partial_x\left(h^{3/2}\partial_x\bar{u}\right)}$$

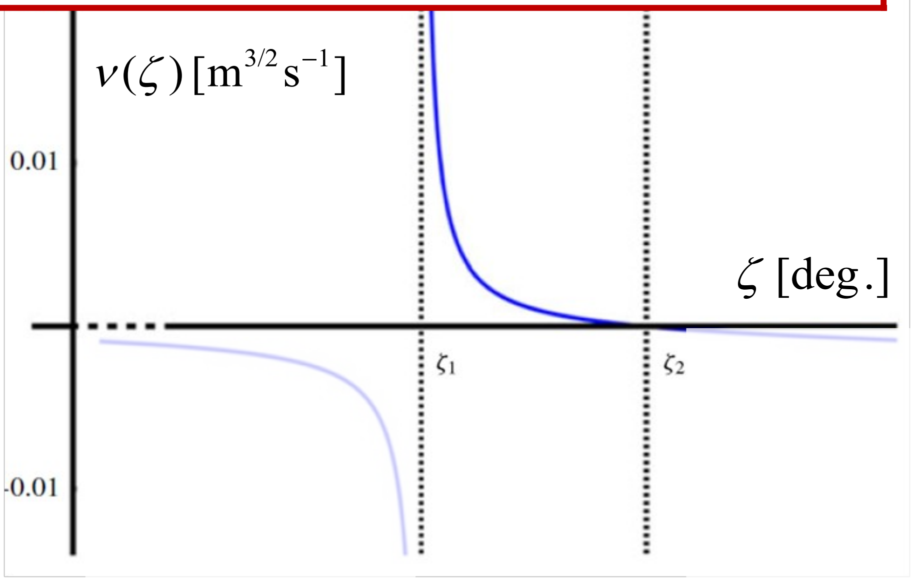
It plays a key role in preventing the waves from developing infinitely steep fronts.

This effective viscosity arises from the *normal stresses* in the granular sheet.

Gray and Edwards, J. Fluid Mech. **755** (2014)

$$\nu = \nu(\zeta) = \frac{2\sqrt{g}}{9}\left(\frac{L}{\beta}\right)\frac{\gamma(\zeta)\sin\zeta}{\sqrt{\cos\zeta}}$$

$$\text{where} \quad \gamma(\zeta) = \frac{\tan\zeta_2 - \tan\zeta}{\tan\zeta - \tan\zeta_1}$$



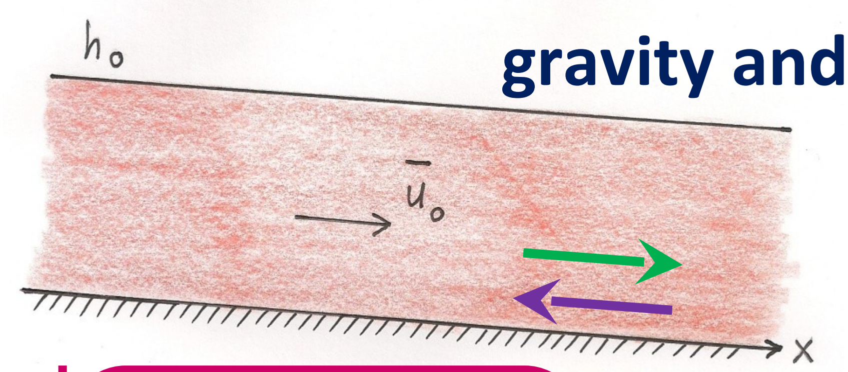
Basic solution: steady uniform flow

$$\cancel{\partial_t\left(h\bar{u}\right)}+\cancel{\partial_x\left(h\bar{u}^2\right)} =$$

$$gh\sin\zeta-\mu(h,\bar{u})gh\cos\zeta-\tfrac{1}{2}\cancel{\partial_x\left(gh^2\cos\zeta\right)}+\nu\,\cancel{\partial_x\left(h^{3/2}\partial_x\bar{u}\right)}$$

balance between the forces of gravity and friction:

$$\mu\left(h,\bar{u}\right)=\tan(\zeta)$$



$$\tan\zeta_1+\frac{\tan\zeta_2-\tan\zeta_1}{1+(\beta/L)\dfrac{h^{3/2}}{\bar{u}}\sqrt{g\cos\zeta}} = \tan\zeta$$

$$\boxed{\begin{aligned}\mathrm{Fr} &= \frac{\bar{u}_0}{\sqrt{h_0\,g\cos\zeta}}\\ &= \frac{\beta}{L\gamma(\zeta)}h\end{aligned}}$$

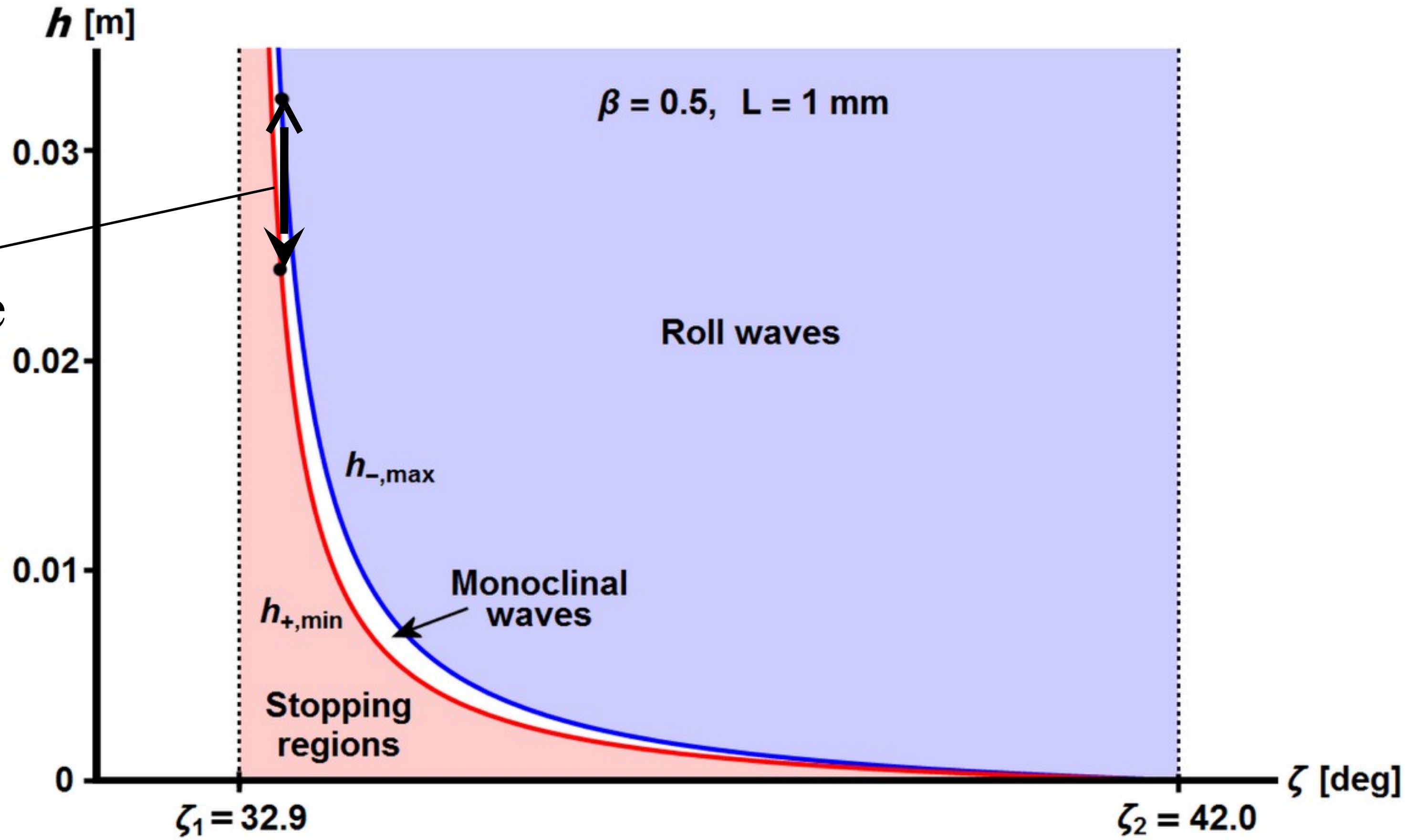
$$\Rightarrow\quad \bar{u}_0=\frac{\beta\sqrt{g\cos\zeta}}{\mathcal{L}\;\gamma(\zeta)}\,h_0^{3/2}$$

We expect monoclinal waves in the range:

Razis, Kanellopoulos & van der Weele, J. Fluid Mech. **843** (2018)

$$\boxed{h_{+,\min}(\zeta)} \quad L\gamma(\zeta) \;\le\; h \;\le\; \frac{2L}{3\beta}\gamma(\zeta) \quad \boxed{h_{-,\max}(\zeta)}$$

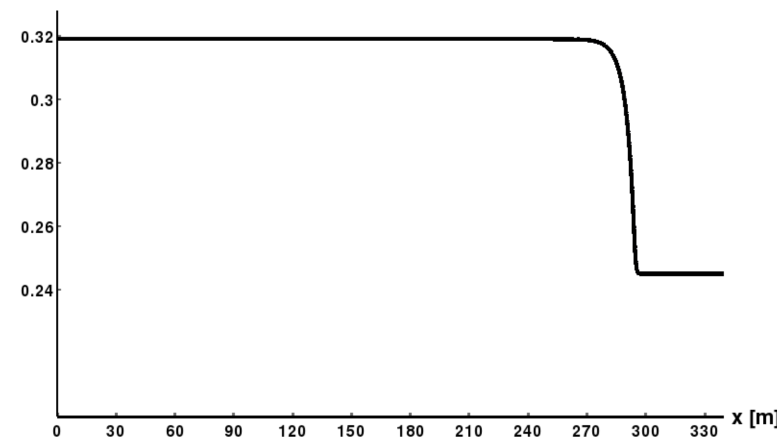
maximal amplitude $\Delta h(\zeta)$



"The proof of the pudding is in the eating"

Numerical experiment

$\boldsymbol{h}\,(\boldsymbol{x},\boldsymbol{t})$ [m]



$\boldsymbol{\bar{u}}\,(\boldsymbol{x},\boldsymbol{t})$ [m/s]

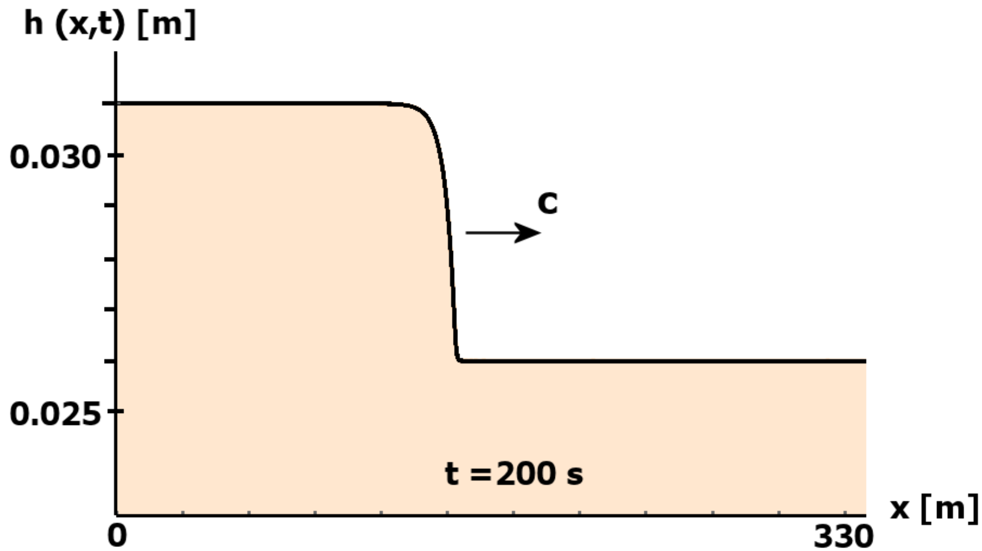
Initial condition (t = 0 s):

we ... eld

→ t ... ly,

producing a genuine monoclinal wave

First-ever observation of a granular monoclinal wave



J. Fluid Mech. (2018), *vol.* 843, *pp.* 810–846.
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The granular monoclinal wave

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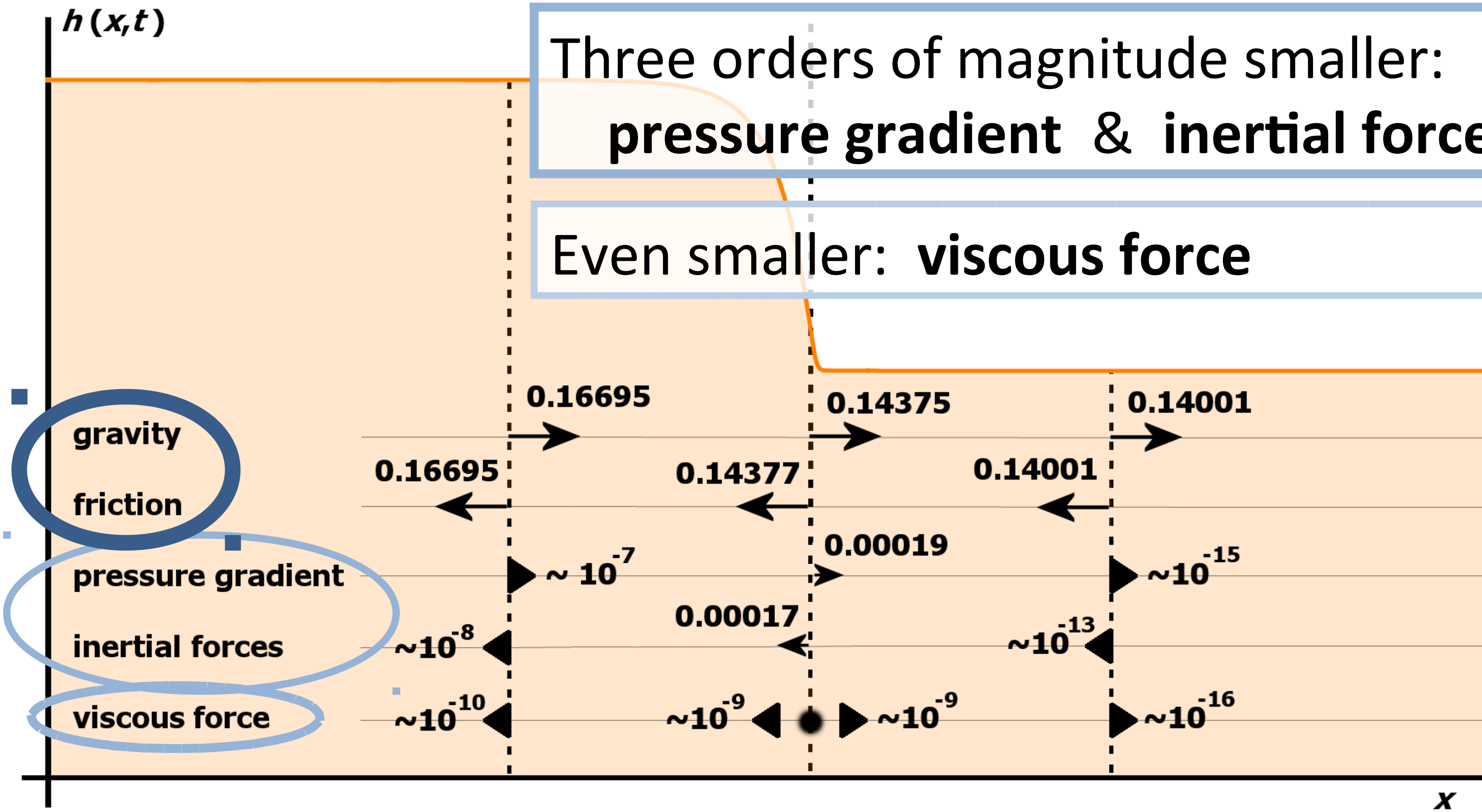
Balance of forces in the monoclinal wave

Main players: **gravity** & **friction**

Three orders of magnitude smaller:
pressure gradient & **inertial forces**

Even smaller: **viscous force**



Speed of the shock : $c = \dfrac{h_- \bar{u}_- - h_+ \bar{u}_+}{h_- - h_+}$

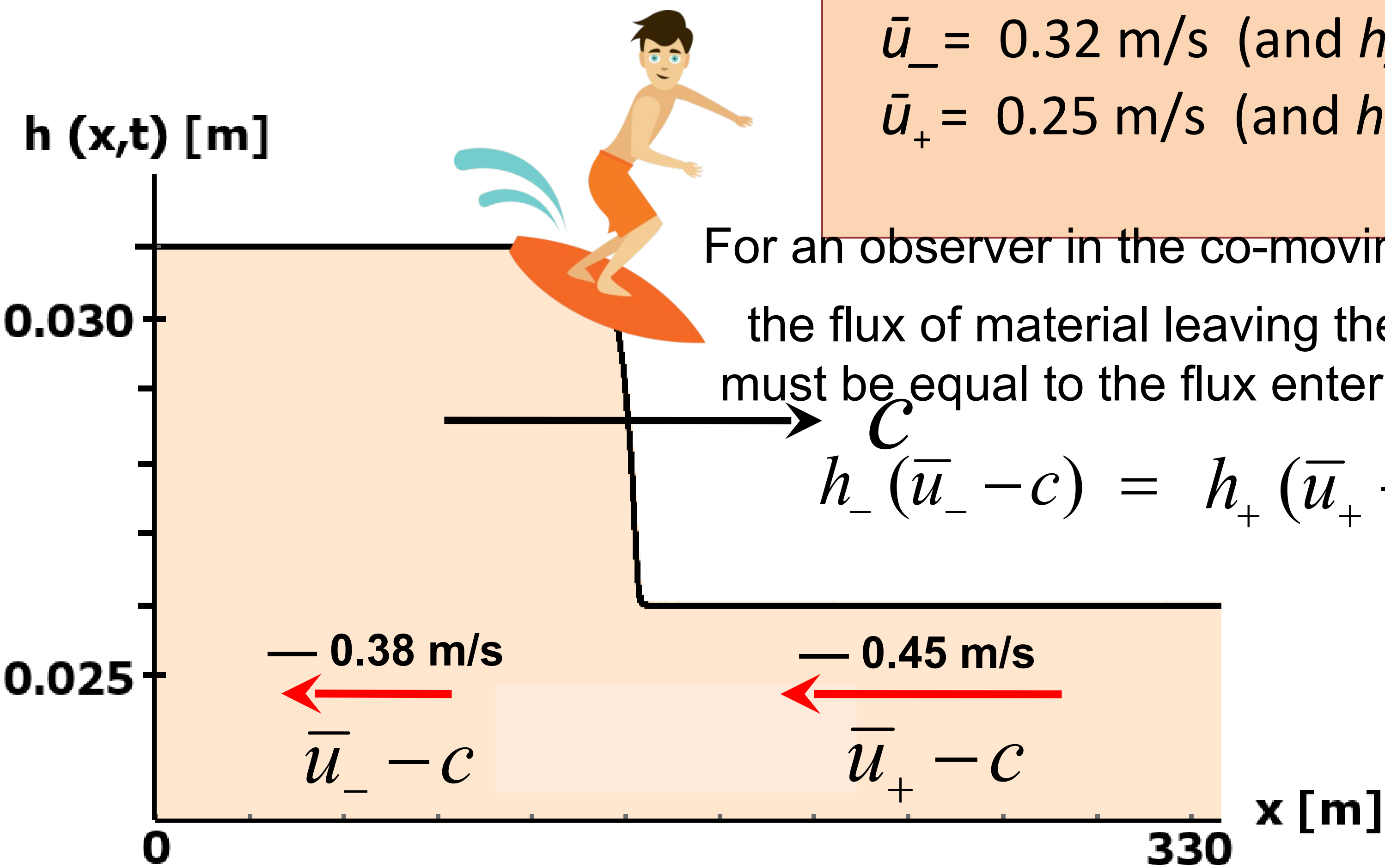
Here: c = 0.70 m/s,

$\bar{u}_-$ = 0.32 m/s (and h_- = 0.032 m),

$\bar{u}_+$ = 0.25 m/s (and h_+ = 0.026 m).

For an observer in the co-moving frame the flux of material leaving the shock zone must be equal to the flux entering the shock:

$$h_- (\bar{u}_- - c) = h_+ (\bar{u}_+ - c)$$



Travelling wave analysis

We introduce the travelling-wave variable

$$\xi = x - ct$$

and will be concerned with solutions of the form

$$h(x,t) = h(x - ct) = h(\xi)$$

$$\overline{u}(x,t) = \overline{u}(x - ct) = \overline{u}(\xi)$$

Then the mass conservation becomes:

$$-c\frac{dh}{d\xi} + \frac{d}{d\xi}\left(h\overline{u}\right) = 0$$

or: $-c\,h' + \left(h\overline{u}\right)' = 0$

This can be integrated immediately:

$$-c\,h' + \left(h\bar{u}\right)' = 0 \quad \Rightarrow \quad -ch + h\bar{u} = \boxed{-K}$$

integration constant

$$K = h\left(c - \bar{u}\right)$$

constant flux observed in the co-moving frame

With this ($\bar{u} = c - K\,h^{-1}$ and hence $\bar{u}' = K\,h^{-2}h'$, etc.) we can eliminate $\bar{u}$ and its derivatives from the momentum balance:

$$\frac{\nu K}{h^{3/2}} h'' - \frac{\nu K}{2 h^{5/2}} (h')^2 + \left(\frac{K}{h^3} - g \cos \zeta \right) h'$$

$$+ g \sin \zeta - \mu(h) g \cos \zeta = 0$$

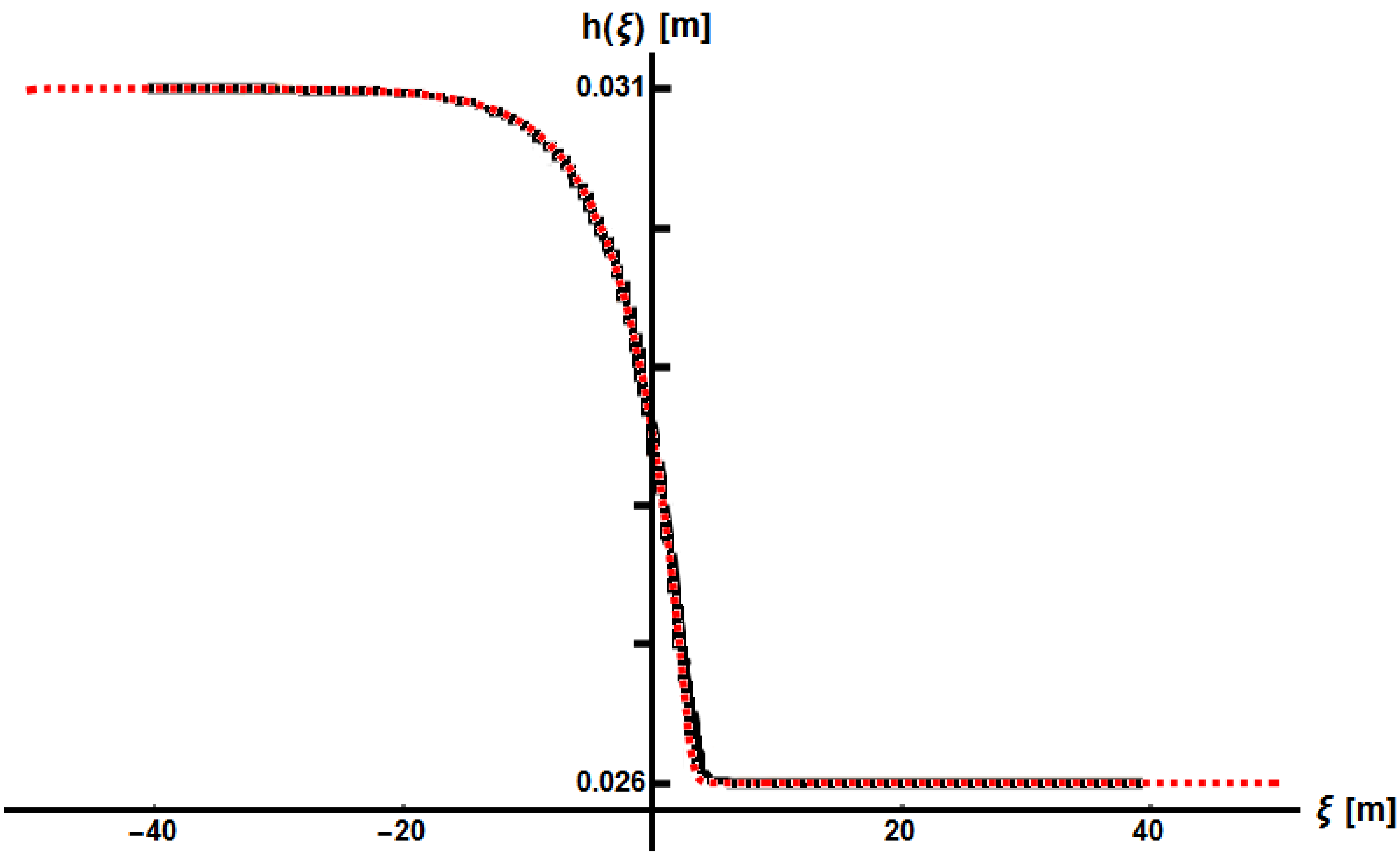
One single second-order ODE for $h(\xi)$!!

Note that with $\overline{u} = c - K h^{-1}$ also the friction coefficient $\mu(h,\bar{u})$ has become a function of ***h*** alone:

$$\mu(h) = \tan(\zeta_1) + \frac{\tan(\zeta_2) - \tan(\zeta_1)}{1 + \beta L^{-1} \sqrt{g \cos(\zeta)}\, h^{5/2} / (ch - K)}$$

This ODE (with the proper boundary conditions) gives ***all*** granular waveforms travelling at constant speed.

In particular also the **monoclinal wave**:



Dimensionless form of the ODE

By measuring all length scales (h and ξ) in terms of the thickness $h\_$ of the incoming sheet, and c in terms of the corresponding velocity $\bar{u}\_$, we obtain:

$$\frac{d^2\tilde{h}}{d\tilde{\xi}^2} - \frac{1}{2\tilde{h}}\left(\frac{d\tilde{h}}{d\tilde{\xi}}\right)^2 + \frac{R_{in}\tilde{h}^{3/2}}{F_{in}^2(\tilde{c}-1)}\left[\left(\frac{F_{in}^2(\tilde{c}-1)^2}{\tilde{h}^3} - 1\right)\frac{d\tilde{h}}{d\tilde{\xi}} + \tan\zeta - \mu(\tilde{h})\right] = 0$$

where R_{in} and F_{in} are the Reynolds and Froude number of the incoming sheet:

$$\boxed{\tilde{h} = \frac{h}{h\_}} \quad \boxed{\tilde{\xi} = \frac{\xi}{h\_}}$$

$$R_{in} = \frac{\bar{u}\_ h\_^{1/2}}{\nu(\zeta)} \ , \qquad F_{in} = \frac{\bar{u}\_}{\sqrt{h\_ g \cos\zeta}}$$

$$\boxed{\tilde{c} = \frac{c}{\bar{u}\_} \propto c\, h\_^{-3/2}}$$

Dynamical Systems approach

Our second-order ODE

$$\frac{d^2\tilde{h}}{d\tilde{\xi}^2} - \frac{1}{2\tilde{h}}\left(\frac{d\tilde{h}}{d\tilde{\xi}}\right)^2 + \frac{R_{in}\tilde{h}^{3/2}}{F_{in}^2(\tilde{c}-1)}\left[\left(\frac{F_{in}^2(\tilde{c}-1)^2}{\tilde{h}^3} - 1\right)\frac{d\tilde{h}}{d\tilde{\xi}} + \tan\zeta - \mu(\tilde{h})\right] = 0$$

can also be written as a system of 2 first-order equations:

$$\begin{cases} \dfrac{d\tilde{h}}{d\tilde{\xi}} = \tilde{s} \quad \boxed{= f(\tilde{h},\tilde{s})} \\[2ex] \dfrac{d\tilde{s}}{d\tilde{\xi}} = \dfrac{\tilde{s}^2}{2\tilde{h}} - \dfrac{R_{in}\tilde{h}^{3/2}}{F_{in}^2(\tilde{c}-1)}\left[\left(\dfrac{F_{in}^2(\tilde{c}-1)^2}{\tilde{h}^3} - 1\right)\tilde{s} + \tan\zeta - \mu(\tilde{h})\right] \quad \boxed{= g(\tilde{h},\tilde{s})} \end{cases}$$

Fixed points:

fixed points correspond to flat regions of the flow

$$\begin{cases} \dfrac{d\tilde{h}}{d\tilde{\xi}} = 0 & \longrightarrow \quad f(\tilde{h},\tilde{s}) = \tilde{s} = 0 \\ \dfrac{d\tilde{s}}{d\tilde{\xi}} = 0 & \longrightarrow \quad g(\tilde{h},\tilde{s}) = g(\tilde{h},0) = 0 \end{cases}$$

which translates to $\mu(\tilde{h}) = \tan\zeta$

Of course!

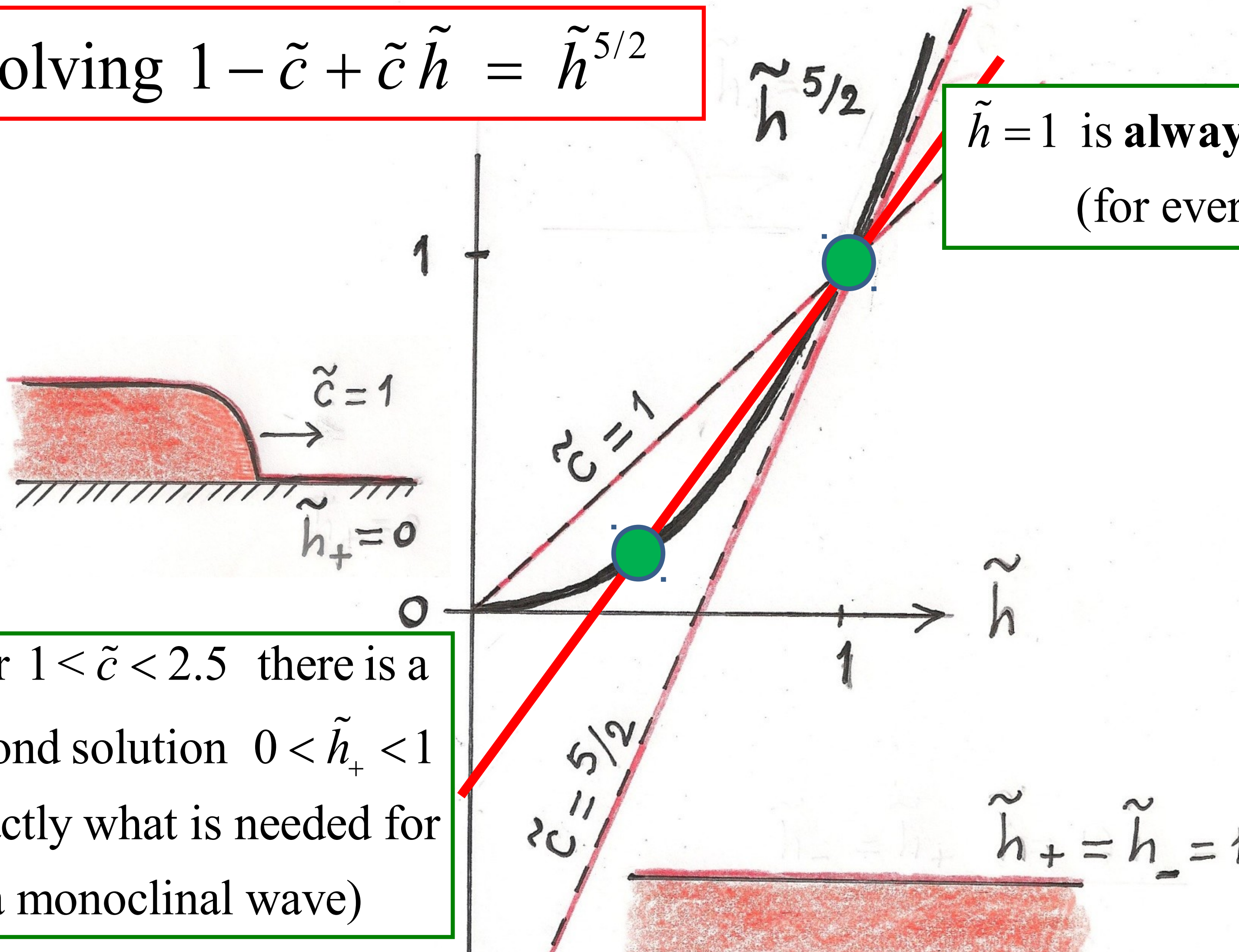
(the familiar balance between friction and gravity in uniform flow regions),

or equivalently:

$$1 - \tilde{c} + \tilde{c}\,\tilde{h} = \tilde{h}^{5/2}$$

Solving $1-\tilde{c}+\tilde{c}\,\tilde{h} \;=\; \tilde{h}^{5/2}$

$\tilde{h}=1$ is **always** a solution (for every $\tilde{c}$)



For $1<\tilde{c}<2.5$ there is a second solution $0<\tilde{h}_+<1$ (exactly what is needed for a monoclinal wave)

$\Rightarrow$ For $1<\tilde{c}<2.5$ the system has *two* fixed points:

$$(\tilde{h},\tilde{s}) \;=\; (\tilde{h}_+,0) \;\text{ and }\; (\tilde{h},\tilde{s}) \;=\; (\tilde{h}_-,0) \;=\; (1,0)$$

Stability of the fixed points

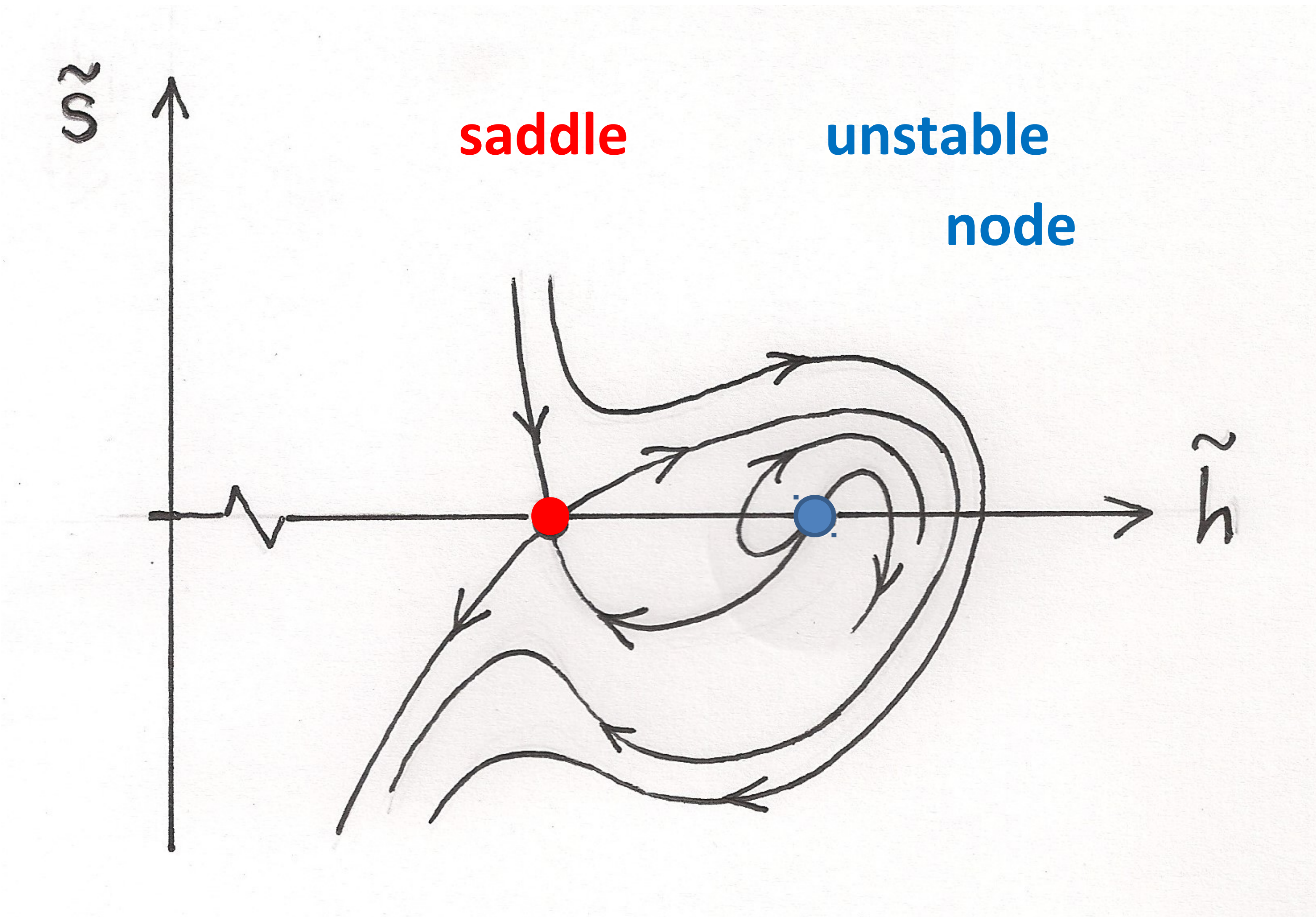
is determined by the eigenvalues of the Jacobian matrix at the fixed points:

$$J = \begin{pmatrix} \dfrac{\partial f(\tilde{h},\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial f(\tilde{h},\tilde{s})}{\partial \tilde{s}} \\ \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{s}} \end{pmatrix}_{(\tilde{h}_\pm,0)} = \begin{pmatrix} 0 & 1 \\ \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{h}} & \dfrac{\partial g(\tilde{h},\tilde{s})}{\partial \tilde{s}} \end{pmatrix}_{(\tilde{h}_\pm,0)}$$

For the parameter values considered here we find that:

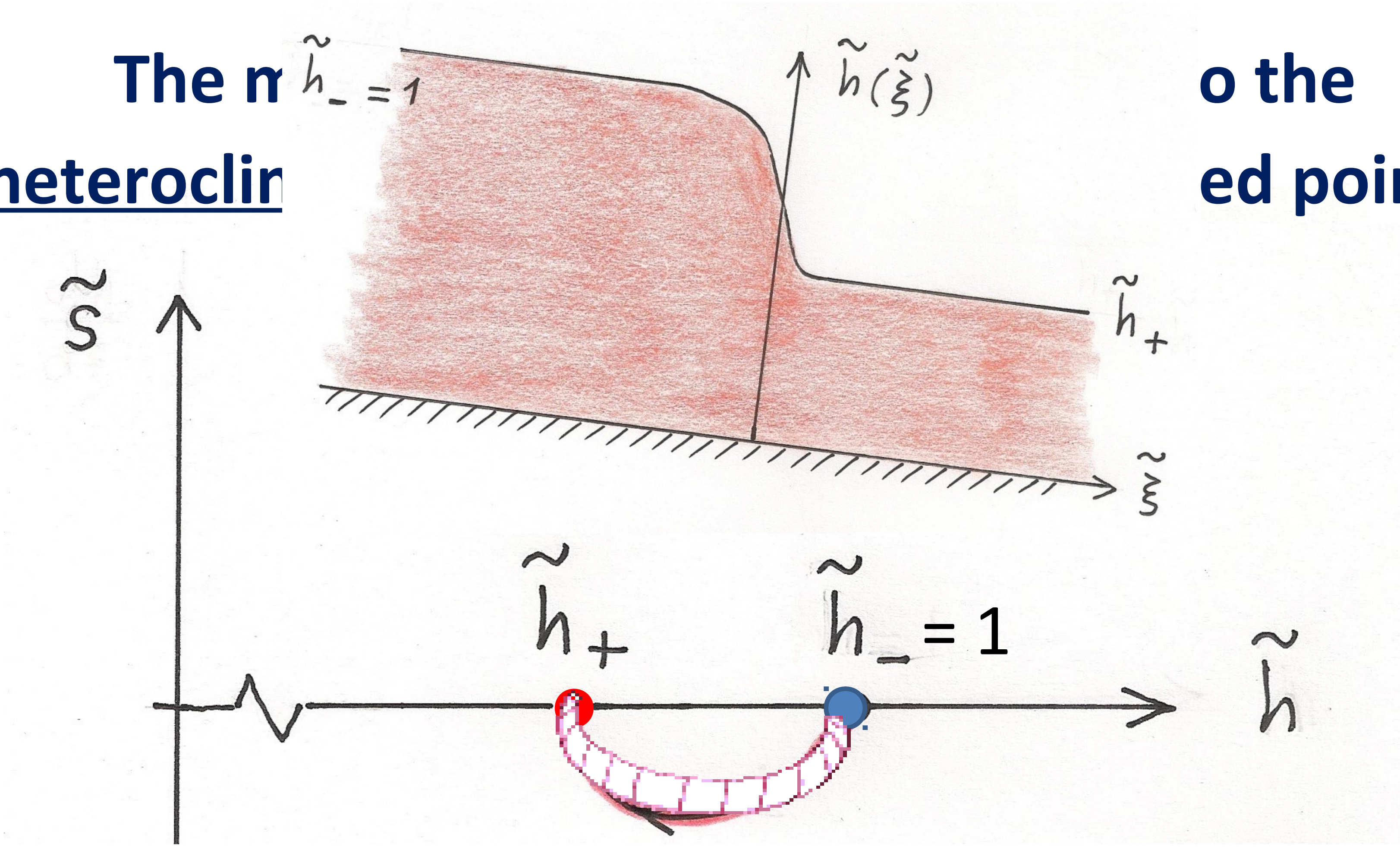
$$\boxed{\begin{array}{l} (\tilde{h}_+,0) \text{ is a saddle point} \\ (\tilde{h}_-,0) = (1,0) \text{ is an unstable node} \end{array}}$$

Phase portrait



The m… o the

heteroclin… ed points:



So we have arrived at a **dynamical systems view** of the granular monoclinal wave

in the co-moving frame,

in dimensionless form.

What more can one ask for!

Concluding remarks

1. **The elusive granular monoclinal wave has been found.**

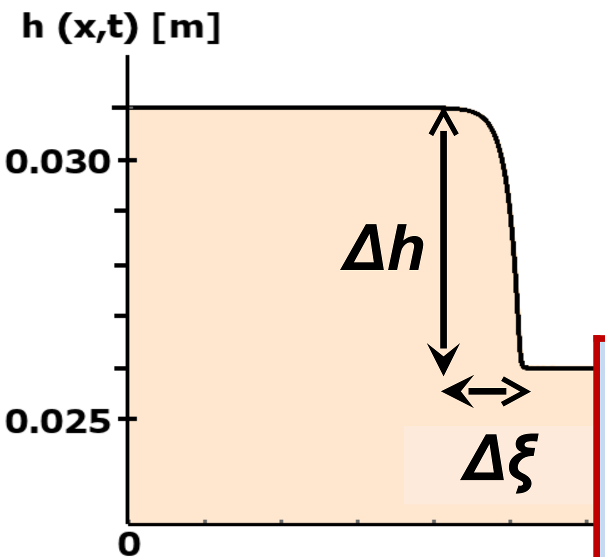
   → **Thus completing the 1-1 correspondence between the "Saint-Venant" waveforms in water and sand.**

Concluding remarks

2. **How feasible will it be to detect a granular monoclinal wave in practice?**

→ **The experimental challenge is:**

«How to maximize the ratio *Δh/Δξ* »



Here: Δh = 0.006 m (= 6 mm)
$\Delta \xi \approx 30$ m

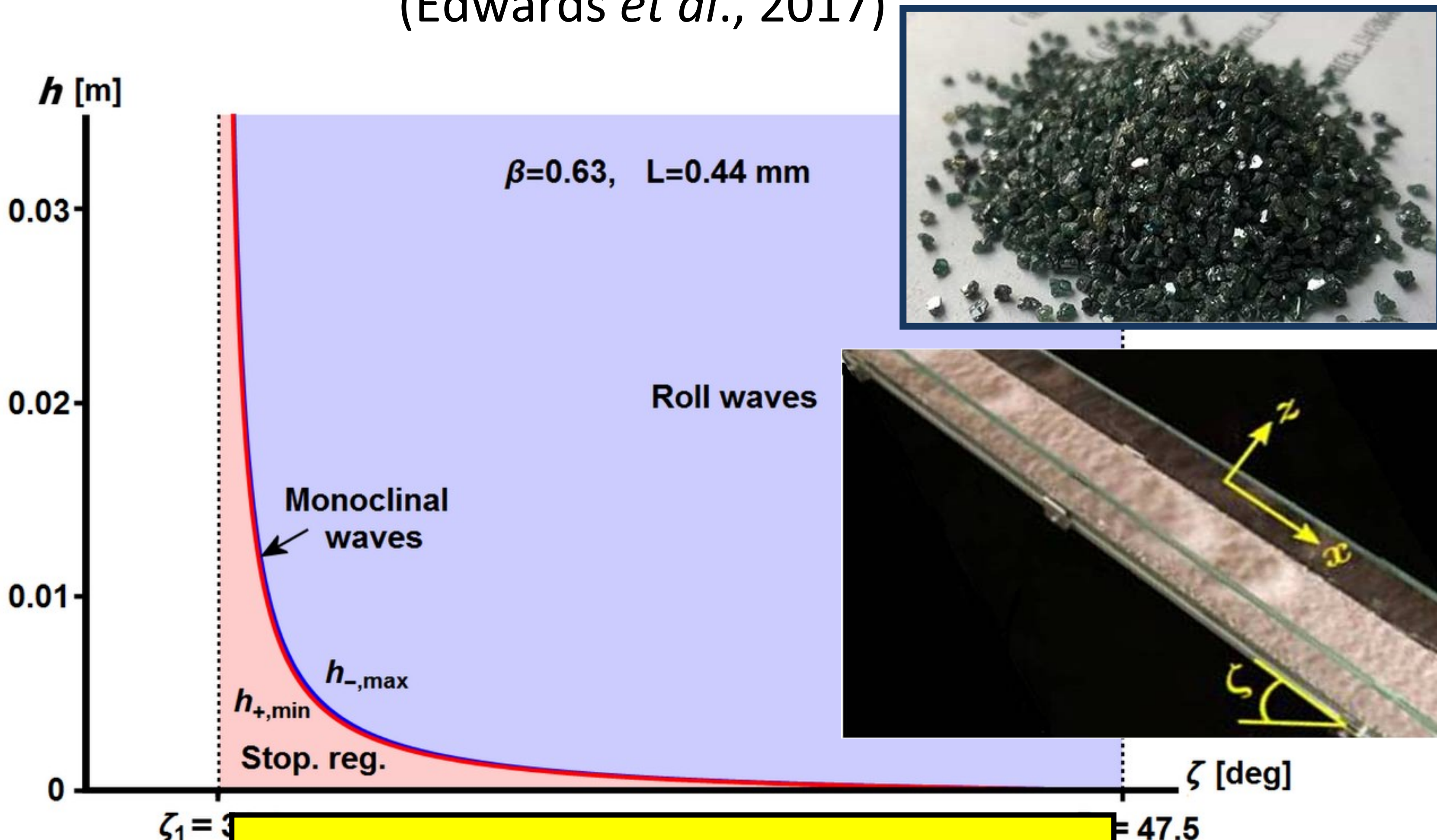
→ ratio $\Delta h/\Delta \xi \approx 2 \cdot 10^{-4}$

This plot is for our "typical parameter values". Let us also have a look at two actual experimental set-ups.

Carborundum particles on the Manchester chute:

β = 0.63, $\boldsymbol{L}$ = 0.44 mm, ζ_1 = 31.1° and ζ_2 = 47.5°

(Edwards *et al.*, 2017)

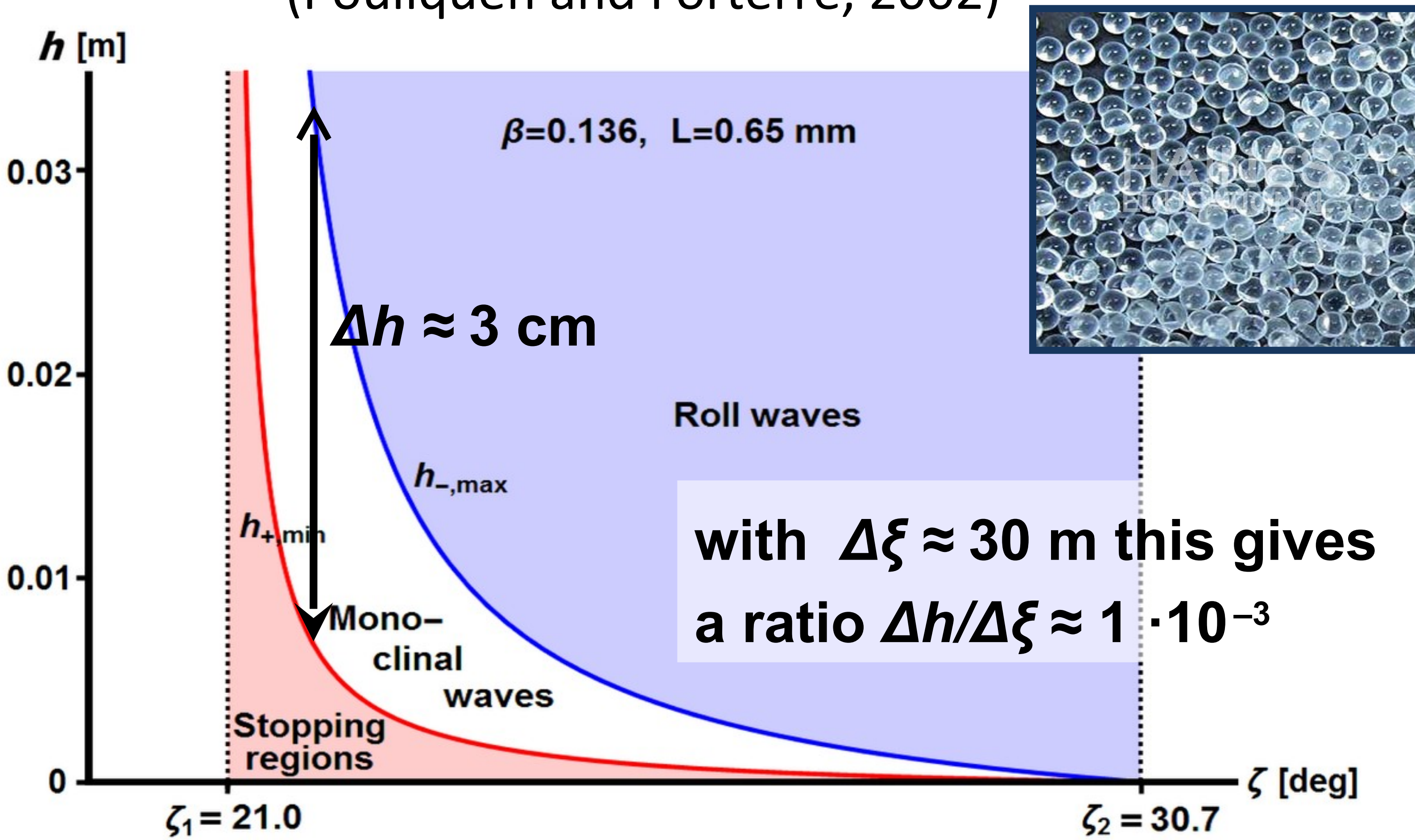


Hardly suitable for spotting monoclinal waves

Smooth glass beads on the chute in Marseille:

β = 0.136, $\boldsymbol{L}$ = 0.65 mm, ζ_1 = 21.0° and ζ_2 = 30.7°

(Pouliquen and Forterre, 2002)



with $\Delta\xi \approx 30$ m this gives a ratio $\Delta h/\Delta\xi \approx 1 \cdot 10^{-3}$

Still challenging, but not impossible

Concluding remarks

3. **Last but not least, the Dynamical Systems approach can also be used to analyze the other waveforms.**

   → **Roll waves**
   → **Rising waves / granular jumps**

Next time, I hope!

Thank you for your attention

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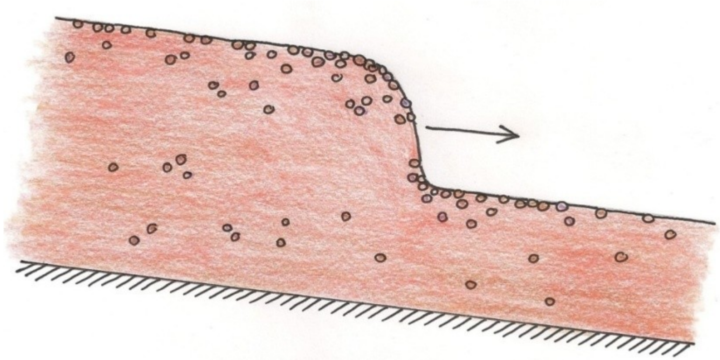


The End

OK or K.O.?